Exactly solvable models through the generalized empty interval method, for multi-species interactions

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Abstract. Multi-species reaction-diffusion systems, with nearest-neighbor interaction on a one-dimensional lattice are considered. Necessary and sufficient constraints on the interaction rates are obtained, that guarantee the closedness of the time evolution equation for $E_n^{\mathbf{a}}(t)$'s, the expectation value of the product of certain linear combination of the number operators on n consecutive sites at time t. The constraints are solved for the single-species left-right-symmetric systems. Also, examples of multi-species system for which the evolution equations of $E_n^{\mathbf{a}}(t)$'s are closed, are given.

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1 Introduction

There is a well-established framework for equilibrium statistical mechanics, but thermal equilibrium is a special case, and there isn't a corresponding straightforward framework for investigating the properties of systems not in equilibrium. There is no general approach to systems far from equilibrium. Different methods have been used to study these models. These include analytical and asymptotic methods, mean-field methods, and large-scale numerical methods. For high-dimensional systems, mean-field techniques give exact or reasonable approximate results. But their results for low-dimensional systems are generally not adequate. So, people are motivated to study stochastic models in low dimensions. Models in low dimensions, should also be in principle easier to study. Exact results for some models on a one-dimensional lattice have been obtained, for example in [1–17].

The term exactly-solvable have been used with different meanings. In [18], a ten-parameter family of reaction-diffusion processes was introduced for which the evolution equation of n-point functions contains only n- or less- point functions. The average particle-number in each site has been obtained exactly for these models. In [19,20], the same method has been used to analyze the above mentioned ten-parameter family model on a finite lattice with boundaries, and in [21] a similar method has been used to study models with next-nearest-neighbor

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interactions. In [22] and [16], integrability means that the N-particle conditional probabilities' S-matrix is factorized into a product of 2-particle S-matrices. Another method which has been used to solve some reaction diffusion models exactly is the empty interval method, and its generalizations.

The empty interval method (EIM) has been used to analyze the one dimensional dynamics of diffusion-limited coalescence [23–26]. Using this method, the probability that n consecutive sites are empty has been calculated. For the cases of finite reaction-rates, some approximate solutions have been obtained. EIM has been also generalized to study the kinetics of the q-state one-dimensional Potts model in the zero-temperature limit [27].

In [28], all the one dimensional reaction-diffusion models with nearest neighbor interactions which can be exactly solved by EIM have been studied. EIM has also been used to study a model with next nearest neighbor interaction [29]. In [30], exactly solvable models through the empty-interval method, for more-than-two-site interactions were studied. In [31], the conventional EIM has been extended to a more generalized form. Using this extended version, a model which can not solved by conventional EIM has been studied.

In this article, we consider systems, in them particles of more than one species interact on a one-dimensional lattice. The interaction is nearest-neighbor. Each site of the lattice, either is empty, or is occupied by one particle. In Section 2, we seek necessary and sufficient conditions on the reaction rates, so that the time evolution

equation for $E^{\mathbf{a}}_{k,n}(t)$ is closed. This quantity is the expectation of the product of a specific linear combination of the number operators (corresponding to different species) at n consecutive sites beginning from the kth site. In Section 3, all single-species left-right symmetric reaction-diffusion systems solvable through the generalized empty-interval method, are classified. In Section 4, multi-species systems are investigated, which are solvable through the generalized empty-interval method, but are effectively single-species. In Section 5, some specific families of two-species systems are investigated, which are exactly solvable through the generalized empty-interval method. Finally, Section 6 is devoted to concluding remarks.

2 Models solvable through the generalized empty-interval method

To fix notations, let us briefly introduce the multi-species reaction-diffusion systems with nearest-neighbor interactions, on a periodic lattice. Let the lattice have $L+1$ sites. The observables of such a system are the operators N_i^α, where i with $1 \leq i \leq L+1$ denotes the site number, and α with $1 \leq \alpha \leq p+1$ denotes the type of the particle. One can regard $\alpha = p+1$ as a vacancy. N_i^α is equal to one, if the site i is occupied by a particle of type α. Otherwise, N_i^α is zero. We also have a constraint

$$s_\alpha N_i^\alpha = 1, \tag{1}$$

where $\mathbf{s}$ is a covector the components of which (s_α's) are all equal to one. The constraint (1), simply says that every site, either is occupied by a particle of one type, or is empty. A representation for these observables is

$$N_i^\alpha := \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes N^\alpha \otimes \underbrace{1 \otimes \cdots \otimes 1}_{L+1-i}, \tag{2}$$

where N^α is a diagonal $(p+1) \times (p+1)$ matrix the only nonzero element of which is the α'th diagonal element, and the operators 1 in the above expression are also $(p+1) \times (p+1)$ matrices. It is seen that the constraint (1) can be written as

$$\mathbf{s} \cdot \mathbf{N} = 1, \tag{3}$$

where $\mathbf{N}$ is a vector the components of which are N^α's. The state of the system is characterized by a vector

$$\mathbf{P} \in \underbrace{\mathbb{V} \otimes \cdots \otimes \mathbb{V}}_{L+1}, \tag{4}$$

where $\mathbb{V}$ is a $(p+1)$-dimensional vector space. All the elements of the vector $\mathbf{P}$ are nonnegative, and

$$\mathbf{S} \cdot \mathbf{P} = 1. \tag{5}$$

Here $\mathbf{S}$ is the tensor-product of $L+1$ covectors $\mathbf{s}$.

As the number operators N_i^α are zero or one (and hence idempotent), the most general observable of such a system is the product of some of these number operators, or a sum of such terms.

The evolution of the state of the system is given by

$$\dot{\mathbf{P}} = \mathcal{H}\,\mathbf{P}, \tag{6}$$

where the Hamiltonian $\mathcal{H}$ is stochastic, by which it is meant that its nondiagonal elements are nonnegative and

$$\mathbf{S}\,\mathcal{H} = 0. \tag{7}$$

The interaction is nearest-neighbor, if the Hamiltonian is of the form

$$\mathcal{H} = \sum_{i=1}^{L+1} H_{i,i+1}, \tag{8}$$

where

$$H_{i,i+1} := \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes H \otimes \underbrace{1 \otimes \cdots \otimes 1}_{L-i}. \tag{9}$$

(It has been assumed that the sites of the system are identical, that is, the system is translation-invariant. Otherwise H in the right-hand side of (9) would depend on i.) The two-site Hamiltonian H is stochastic, that is, its nondiagonal elements are nonnegative, and the sum of the elements of each of its columns vanishes:

$$(\mathbf{s} \otimes \mathbf{s})H = 0. \tag{10}$$

Now consider a certain class of such observables, namely

$$\mathcal{E}^{\mathbf{a}}_{k,n} := \prod_{l=k}^{k+n-1} (\mathbf{a} \cdot \mathbf{N}_l), \tag{11}$$

where $\mathbf{a}$ is a specific $(p+1)$-dimensional covector, and $\mathbf{N}_i$ is a vector the components of which are the operators N_i^α. We want to find criteria for H, so that the evolutions of the expectations of $\mathcal{E}^{\mathbf{a}}_{k,n}$'s are closed, that is the time-derivative of their expectation is expressible in terms of the expectations of $\mathcal{E}^{\mathbf{a}}_{k,n}$'s themselves. Denoting the expectations of these observables by $E^{\mathbf{a}}_{k,n}$,

$$E^{\mathbf{a}}_{k,n} := \mathbf{S}\,\mathcal{E}^{\mathbf{a}}_{k,n}\mathbf{P}, \tag{12}$$

we have

$$\begin{aligned}
\dot{E}^{\mathbf{a}}_{k,n} &= \mathbf{S}\,\mathcal{E}^{\mathbf{a}}_{k,n}\mathcal{H}\,\mathbf{P}, \\
&= \mathbf{S}\,\mathcal{E}^{\mathbf{a}}_{k,n}H_{k-1,k}\,\mathbf{P} \\
&\quad + \sum_{l=1}^{n-1} \mathbf{S}\,\mathcal{E}^{\mathbf{a}}_{k,n}H_{k-1+l,k+l}\,\mathbf{P} \\
&\quad + \mathbf{S}\,\mathcal{E}^{\mathbf{a}}_{k,n}H_{k+n-1,k+n}\,\mathbf{P}.
\end{aligned} \tag{13}$$

From this, and using (3), it is seen that the necessary and sufficient conditions that the left-hand side be expressible in terms of $E^{\mathbf{a}}_{k,n}$'s are

$$\begin{aligned}
(\mathbf{s} \otimes \mathbf{s})[(\mathbf{s} \cdot \mathbf{N}) \otimes (\mathbf{a} \cdot \mathbf{N})]H &= \mu_{\mathrm{L}}(\mathbf{s} \otimes \mathbf{s})[(\mathbf{s} \cdot \mathbf{N}) \otimes (\mathbf{a} \cdot \mathbf{N})] \\
&+ \theta_{\mathrm{L}}(\mathbf{s} \otimes \mathbf{s})[(\mathbf{s} \cdot \mathbf{N}) \otimes (\mathbf{s} \cdot \mathbf{N})] \\
&+ \nu_{\mathrm{L}}(\mathbf{s} \otimes \mathbf{s})[(\mathbf{a} \cdot \mathbf{N}) \otimes (\mathbf{a} \cdot \mathbf{N})], \quad (14)
\end{aligned}$$

$$(\mathbf{s} \otimes \mathbf{s})[(\mathbf{a} \cdot \mathbf{N}) \otimes (\mathbf{a} \cdot \mathbf{N})]H = \lambda(\mathbf{s} \otimes \mathbf{s})[(\mathbf{a} \cdot \mathbf{N}) \otimes (\mathbf{a} \cdot \mathbf{N})], \tag{15}$$

$$(\mathbf{s} \otimes \mathbf{s})[(\mathbf{a} \cdot \mathbf{N}) \otimes (\mathbf{s} \cdot \mathbf{N})]H = \mu_{\mathrm{R}}(\mathbf{s} \otimes \mathbf{s})[(\mathbf{a} \cdot \mathbf{N}) \otimes (\mathbf{s} \cdot \mathbf{N})]$$
$$+ \theta_{\mathrm{R}}(\mathbf{s} \otimes \mathbf{s})[(\mathbf{s} \cdot \mathbf{N}) \otimes (\mathbf{s} \cdot \mathbf{N})]$$
$$+ \nu_{\mathrm{R}}(\mathbf{s} \otimes \mathbf{s})[(\mathbf{a} \cdot \mathbf{N}) \otimes (\mathbf{a} \cdot \mathbf{N})], \quad (16)$$

for some arbitrary numbers λ, $\mu_{\mathrm{L},R}$, $\theta_{\mathrm{L},R}$, and $\nu_{\mathrm{L},R}$. Using the identity

$$\mathbf{s}(\mathbf{b} \cdot \mathbf{N}) = \mathbf{b}, \tag{17}$$

for an arbitrary covector $\mathbf{b}$, one arrives at

$$(\mathbf{s} \otimes \mathbf{a})H = \mu_{\mathrm{L}}(\mathbf{s} \otimes \mathbf{a}) + \theta_{\mathrm{L}}(\mathbf{s} \otimes \mathbf{s}) + \nu_{\mathrm{L}}(\mathbf{a} \otimes \mathbf{a}),$$
$$(\mathbf{a} \otimes \mathbf{a})H = \lambda(\mathbf{a} \otimes \mathbf{a}),$$
$$(\mathbf{a} \otimes \mathbf{s})H = \mu_{\mathrm{R}}(\mathbf{a} \otimes \mathbf{s}) + \theta_{\mathrm{R}}(\mathbf{s} \otimes \mathbf{s}) + \nu_{\mathrm{R}}(\mathbf{a} \otimes \mathbf{a}). \tag{18}$$

The Hamiltonian H should of course satisfy (10) as well, and its nondiagonal elements should be nonnegative. In this case, the evolution equation for $E_{k,n}^{\mathbf{a}}$, for $0 < n < L + 1$, becomes

$$\dot{E}_{k,n}^{\mathbf{a}} = \theta_{\mathrm{L}} E_{k+1,n-1}^{\mathbf{a}} + \nu_{\mathrm{L}} E_{k-1,n+1}^{\mathbf{a}} + \theta_{\mathrm{R}} E_{k,n-1}^{\mathbf{a}}$$
$$+ \nu_{\mathrm{R}} E_{k,n+1}^{\mathbf{a}} + [\mu_{\mathrm{L}} + (n-1)\lambda + \mu_{\mathrm{R}}] E_{k,n}^{\mathbf{a}}. \quad (19)$$

For $n = L + 1$, one has

$$\dot{E}_{1,L+1}^{\mathbf{a}} = (L+1)\lambda E_{1,L+1}^{\mathbf{a}}. \tag{20}$$

For $n = 0$, from (5) we have the boundary condition

$$E_{k,0}^{\mathbf{a}} = 1. \tag{21}$$

Equations (19, 20), and the boundary condition (21) are a closed set of evolution equations for $E_{i,j}^{\mathbf{a}}$'s. These equations are quite similar to those obtained in [28]. In fact, the case there is a special case of what considered here, with $p = 1$ and $\mathbf{a} = (0, 1)$. Although the criterion for the closedness of the evolution equations does depend on p and $\mathbf{a}$, the evolution equations for $E_{i,j}^{\mathbf{a}}$'s do not depend on these. In [28], situations were considered in which λ was zero, so that the evolution of the block comes solely from its ends. This makes solving the evolution equations easier. Finally, since the system under consideration is translationally-symmetric, if the initial condition is translationally-invariant, the state of the system would remain translationally-invariant at all times. In this case, $E_{k,n}^{\mathbf{a}}$ does not depend on k, and one would have

$$\dot{E}_n^{\mathbf{a}} = (\theta_{\mathrm{L}} + \theta_{\mathrm{R}}) E_{n-1}^{\mathbf{a}} + (\nu_{\mathrm{L}} + \nu_{\mathrm{R}}) E_{n+1}^{\mathbf{a}}$$
$$+ [\mu_{\mathrm{L}} + (n-1)\lambda + \mu_{\mathrm{R}}] E_n^{\mathbf{a}}, \quad 0 < n < L + 1,$$
$$\dot{E}_{L+1}^{\mathbf{a}} = (L+1)\lambda E_{L+1}^{\mathbf{a}},$$
$$E_0^{\mathbf{a}} = 1. \tag{22}$$

The system is left-right symmetric, iff the Hamiltonian is invariant under permutation:

$$\Pi \, H \, \Pi = H, \tag{23}$$

where Π is the permutation matrix:

$$\Pi(\mathbf{u} \otimes \mathbf{v}) = \mathbf{v} \otimes \mathbf{u}. \tag{24}$$

It is easily seen that if H satisfies (23) and (18), then

$$\mu_{\mathrm{L}} = \mu_{\mathrm{R}} = \mu, \quad \nu_{\mathrm{L}} = \nu_{\mathrm{R}} = \nu, \quad \theta_{\mathrm{L}} = \theta_{\mathrm{R}} = \theta. \tag{25}$$

3 Classification of the single-species left-right-symmetric reaction-diffusion systems, which are solvable through the generalized empty-interval method

For a single-species system, the vector space $\mathbb{V}$ is two-dimensional. Take a covector $\mathbf{a}$, which is not a multiple of $\mathbf{s}$. (The case $\mathbf{a}$ proportional to $\mathbf{s}$ is trivial, as $\mathbf{s} \cdot \mathbf{N} = 1$). The set $B := \{\mathbf{a} \otimes \mathbf{a}, \mathbf{a} \otimes \mathbf{s}, \mathbf{s} \otimes \mathbf{a}, \mathbf{s} \otimes \mathbf{s}\}$, is a basis for $\mathbb{V} \otimes \mathbb{V}$. If H satisfies (18) (and of course (10)), then one has the matrix elements of the Hamiltonian in this basis. However, not every Hamiltonian in this form represents a stochastic system. The nondiagonal elements of the Hamiltonian in the physical (ordinary) basis should be nonnegative. So, for the single-species systems, the task of classifying the systems solvable through the generalized empty-interval method, reduces to finding the matrix elements of the Hamiltonian in the physical basis (in terms of the covector $\mathbf{a}$, and the scalars λ, μ, ν, and θ); and imposing the criterion that the nondiagonal elements of H in the physical basis be nonnegative. Taking the covector $\mathbf{a}$ like

$$\mathbf{a} = (a_1, a_2), \tag{26}$$

and noting that $a_1 \neq a_2$ (otherwise $\mathbf{a}$ would be proportional to $\mathbf{s}$), and that one can rescale $\mathbf{a}$ without changing the conditions (18), it is seen that one can take

$$a_1 = \xi + 1, \quad a_2 = \xi - 1, \tag{27}$$

without loss of generality. Then, imposing the criterion that the nondiagonal elements of the Hamiltonian are non-negative, and assuming left-right symmetry, one arrives at the following set of inequalities.

$$(1 + \xi^2)\Lambda + 2\xi^2\mu + 2\xi\theta \geq |2\xi(\Lambda + \mu)|,$$
$$-(1 + \xi^2)(2\mu + \Lambda) - 4\xi\nu - 2\xi\theta \geq |2[(1 + \xi^2)\nu$$
$$+ \xi(\Lambda + 2\mu) + \theta]|,$$
$$-2\mu \geq (1 - \xi^2)\Lambda - 2\xi^2\mu - 2\xi\theta \geq |2[(\xi^2 - 1)\nu + \xi\mu + \theta]|, \tag{28}$$

where

$$\Lambda := -\lambda + 2\xi\nu. \tag{29}$$

We also define

$$\tau := \theta - \xi,$$
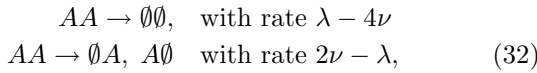$$\Lambda^{\pm} := \Lambda \pm 2\nu. \tag{30}$$

We have to solve the inequalities (28), for a given value of ξ. It is seen that changing the signs of ξ, ν, and θ simultaneously, while keeping the signs of λ, μ, and Λ fixed, does not change the inequalities. So it is sufficient to solve the inequalities for nonnegative ξ.

The detailed calculations can be found in the appendix. The results are the following.
i)

$$\xi = 1, \quad \mu = \theta = 0, \quad \nu \le 0, \quad 4\nu \le \lambda \le 2\nu. \tag{31}$$

The reactions for systems in this class are

$$
\begin{aligned}
AA &\to \emptyset\emptyset, \quad \text{with rate } \lambda - 4\nu \\
AA &\to \emptyset A, \ A\emptyset \quad \text{with rate } 2\nu - \lambda,
\end{aligned}
\tag{32}
$$

for which the evolution equation of $\langle n_1 \cdots n_k \rangle$ is closed, where n_i is the number operator at the site i.
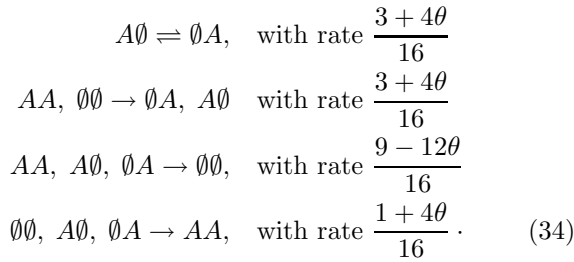ii)

$$0 < \xi < 1, \quad \mu < 0 \ (\mu = -1) \quad (\Lambda^-, \Lambda^+) \text{ inside}$$
$$\text{the tetragon } ABCE, \quad \tau \text{ satisfies (58).} \tag{33}$$

The coordinates of the vertices of the tetragon $ABCE$ are

$$A\left(-\frac{1-\xi}{1+\xi}, 1\right), \quad B\left(1, \frac{3+\xi}{1+\xi}\right), \quad C\left(\frac{3-\xi}{1-\xi}, 1\right),$$
$$E\left(\frac{1+6\xi-3\xi^2}{(1-\xi)(1+3\xi)}, -\frac{1+3\xi^2}{(1-\xi)(1+3\xi)}\right),$$

where the first coordinate is Λ^-, and the second is Λ^+. As an example in this class, take $\xi = 1/2$, and $\Lambda^+ = \Lambda^- = 1$, which lead to a system with following interactions.

$$A\emptyset \rightleftharpoons \emptyset A, \quad \text{with rate } \frac{3+4\theta}{16}$$
$$AA, \ \emptyset\emptyset \to \emptyset A, \ A\emptyset \quad \text{with rate } \frac{3+4\theta}{16}$$
$$AA, \ A\emptyset, \ \emptyset A \to \emptyset\emptyset, \quad \text{with rate } \frac{9-12\theta}{16}$$
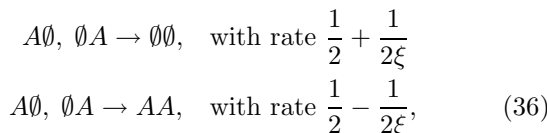$$\emptyset\emptyset, \ A\emptyset, \ \emptyset A \to AA, \quad \text{with rate } \frac{1+4\theta}{16}. \tag{34}$$

For this system, the evolution equation of $\langle [2n_1 - (1/2)] \cdots [2n_k - (1/2)] \rangle$ is closed.
iii)

$$1 < \xi, \quad \mu < 0 \ (\mu = -1), \quad \tau = -\frac{\xi^2+1}{2\xi}, \quad \Lambda^{\pm} = \frac{\xi \pm 1}{\xi}. \tag{35}$$

It is seen that here there is no allowed region for the rates, but a single point (apart from scaling) for each value of ξ. Systems in this class, correspond to the reactions

$$A\emptyset, \ \emptyset A \to \emptyset\emptyset, \quad \text{with rate } \frac{1}{2} + \frac{1}{2\xi}$$
$$A\emptyset, \ \emptyset A \to AA, \quad \text{with rate } \frac{1}{2} - \frac{1}{2\xi}, \tag{36}$$

and for them the evolution equation of $\langle (2n_1 + \xi - 1) \cdots (2n_k + \xi - 1) \rangle$ is closed.
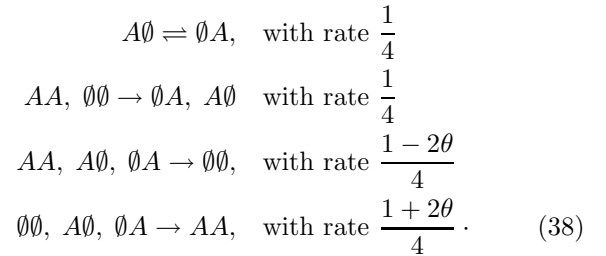iv)

$$\xi = 0, \quad \mu \ne 0 \ (\mu = -1), \quad (\Lambda^-, \Lambda^+) \text{ is inside}$$
$$\text{the square } A_0 B_0 C_0 D_0. \ \tau \text{ satisfies (57).} \tag{37}$$

The vertices of the tetragon $A_0 B_0 C_0 D_0$ are

$$A_0(-1,1), \quad B_0(1,3), \quad C_0(3,1), \quad D_0(1,-1).$$

As an example in this class, take $\Lambda^+ = \Lambda^- = 1$, which lead to a system with following interactions.

$$A\emptyset \rightleftharpoons \emptyset A, \quad \text{with rate } \frac{1}{4}$$
$$AA, \ \emptyset\emptyset \to \emptyset A, \ A\emptyset \quad \text{with rate } \frac{1}{4}$$
$$AA, \ A\emptyset, \ \emptyset A \to \emptyset\emptyset, \quad \text{with rate } \frac{1-2\theta}{4}$$
$$\emptyset\emptyset, \ A\emptyset, \ \emptyset A \to AA, \quad \text{with rate } \frac{1+2\theta}{4}. \tag{38}$$

For this system, the evolution equation of $\langle (2n_1 - 1) \cdots \otimes (2n_k - 1) \rangle$ is closed.
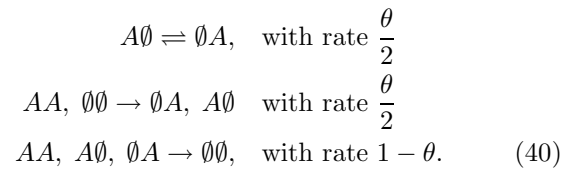v)

$$\xi = 1, \quad \mu \ne 0 \ (\mu = -1), \quad (\Lambda^-, \Lambda^+) \text{ is inside } \mathbb{S}.$$
$$\tau \text{ satisfies (70).} \tag{39}$$

The region $\mathbb{S}$, is the region limited by the lines $A_1 B'$, the horizontal line passing through B', and the line passing through A_1 with the slope -1, containing the point $(1,1)$, where

$$A_1(0,1), \quad B'(0,2).$$

As an example in this class, take $\Lambda^+ = \Lambda^- = 1$, which lead to a system with following interactions.

$$A\emptyset \rightleftharpoons \emptyset A, \quad \text{with rate } \frac{\theta}{2}$$
$$AA, \ \emptyset\emptyset \to \emptyset A, \ A\emptyset \quad \text{with rate } \frac{\theta}{2}$$
$$AA, \ A\emptyset, \ \emptyset A \to \emptyset\emptyset, \quad \text{with rate } 1 - \theta. \tag{40}$$

For this system, the evolution equation of $\langle n_1 \cdots n_k \rangle$ is closed.

4 Effectively single-species systems

For a specific $(p+1)$-dimensional covector, we seek Hamiltonians satisfying (18) with some values of λ, $\mu_{L,R}$, $\nu_{L,R}$, and $\theta_{L,R}$. There are cases, however, where p-species systems are effectively single-species. Suppose we can decompose the states of the system into two subsets 1 and 2. Corresponding to these states, one defines the covectors $\mathbf{E}^1$, and $\mathbf{E}^2$. $\mathbf{E}^1$, for example, is the sum of the covectors

corresponding to the states belonging to the first subset (the microstates of the first state). It is clear that
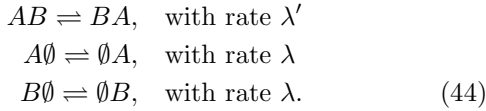
$$\mathbf{E}^1 + \mathbf{E}^2 = \mathbf{s}. \qquad (41)$$

To have an effectively two-state system, the probability that the system goes from one of the microstates of the state i to the state j, should not depend on the microstates. In terms of the Hamiltonian, this means

$$\mathbf{E}^i H = \sum_j \tilde{H}^i{}_j \mathbf{E}^j. \qquad (42)$$

In this case, the system described by the Hamiltonian $\tilde{H}$ is a two-state system. If, moreover, one seeks informations about the original system, which are expressible in terms of only the states 1 and 2 (and not dependent on the microstates) then the system is an effectively two-state system. For example, if the states of a system are white, blue, and red, one can define the states white and colored, provided the probability that the system changes from red to white is the same as that of changing from blue to white. If in addition, we are only interested in probabilities of the system being white or colored, then the system is effectively a two-state system. Obvious generalizations of these systems with nearest-neighbor interactions on a lattice, are systems for which

$$(\mathbf{E}^i \otimes \mathbf{E}^j) H = \sum_{k,l} \tilde{H}^{ij}{}_{kl}(\mathbf{E}^k \otimes \mathbf{E}^l). \qquad (43)$$

An example is a system consisting of two kinds of particles (A and B) diffusing on a lattice:

$$\begin{aligned} AB &\rightleftharpoons BA, \quad \text{with rate } \lambda' \\ A\emptyset &\rightleftharpoons \emptyset A, \quad \text{with rate } \lambda \\ B\emptyset &\rightleftharpoons \emptyset B, \quad \text{with rate } \lambda. \end{aligned} \qquad (44)$$

This system is effectively one-species, as far as one is concerned only with probabilities of finding particles (not of a specific type).

Now let us return to the problem of finding solutions to (18), with a specified covector $\mathbf{a}$. We want to show that if the components of the covector $\mathbf{a}$ take only two values, then the system under consideration is an effectively two-state system (or an effectively single-species system, with the states occupied and empty). If the components of $\mathbf{a}$ take only two values, then one can write $\mathbf{a}$ as

$$\mathbf{a} = a_1 \mathbf{E}^1 + a_2 \mathbf{E}^2, \qquad (45)$$

where $\mathbf{E}^i$'s are covectors with the property that their components are either zero or one, and their sum is equal to $\mathbf{s}$. It is seen that the covectors $\mathbf{s}$ and $\mathbf{a}$ are linear combinations of $\mathbf{E}^1$ and $\mathbf{E}^2$, and *vice versa*. So a result of equations (18, 10) is that $(\mathbf{E}^i \otimes \mathbf{E}^j)H$ is a linear combination of $(\mathbf{E}^k \otimes \mathbf{E}^l)$'s; that is, (43) holds. Also, one notes that

$$\mathbf{a} \cdot \mathbf{N} = a_1 \mathbf{E}^1 \cdot \mathbf{N} + a_2 \mathbf{E}^2 \cdot \mathbf{N}, \qquad (46)$$
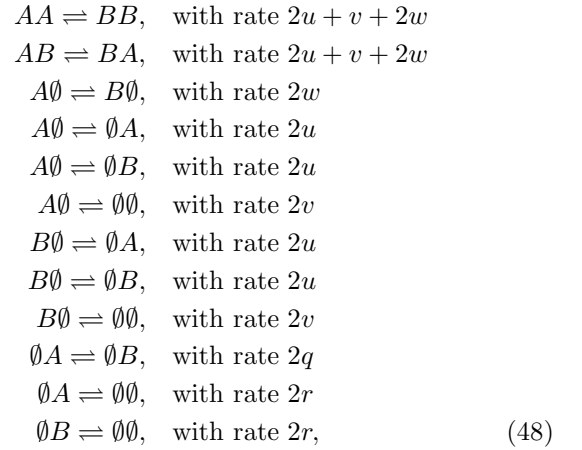
which means that the information we seek involves the probabilities corresponding to only the subsets 1 and 2. So, multi-species systems solvable through the generalized empty-interval method, with covectors the components of which take only two values, are effectively single-species.

5 Some two-species examples

The states of a two-species system on each site of the lattice can be represented by A, B, and $\emptyset$, the latter being a vacancy. As the first example, consider a system for which the Hamiltonian is symmetric, which means the rate of each reaction is equal to the rate of its reverse reaction. Also let $\lambda = 0$ in (18). For the three-dimensional covector $\mathbf{a}$, take the choice

$$\mathbf{a} = (1, -1, 0). \qquad (47)$$

One can then solve (18) and (10) for H and $\mu_{\mathrm{L,R}}$, $\nu_{\mathrm{L,R}}$, and $\theta_{\mathrm{L,R}}$. Also the rates (the nondiagonal elements of H) should be nonnegative. These constraints lead to a system with the following reaction rates:

$$\begin{aligned} AA &\rightleftharpoons BB, \quad \text{with rate } 2u + v + 2w \\ AB &\rightleftharpoons BA, \quad \text{with rate } 2u + v + 2w \\ A\emptyset &\rightleftharpoons B\emptyset, \quad \text{with rate } 2w \\ A\emptyset &\rightleftharpoons \emptyset A, \quad \text{with rate } 2u \\ A\emptyset &\rightleftharpoons \emptyset B, \quad \text{with rate } 2u \\ A\emptyset &\rightleftharpoons \emptyset\emptyset, \quad \text{with rate } 2v \\ B\emptyset &\rightleftharpoons \emptyset A, \quad \text{with rate } 2u \\ B\emptyset &\rightleftharpoons \emptyset B, \quad \text{with rate } 2u \\ B\emptyset &\rightleftharpoons \emptyset\emptyset, \quad \text{with rate } 2v \\ \emptyset A &\rightleftharpoons \emptyset B, \quad \text{with rate } 2q \\ \emptyset A &\rightleftharpoons \emptyset\emptyset, \quad \text{with rate } 2r \\ \emptyset B &\rightleftharpoons \emptyset\emptyset, \quad \text{with rate } 2r, \end{aligned} \qquad (48)$$

with the condition

$$r + 2q = v + 2w. \qquad (49)$$

For these rates,

$$\begin{aligned} \mu_{\mathrm{L}} &= \mu_{\mathrm{R}} = -(4u + 2v + 4w), \\ \nu_{\mathrm{L}} &= \nu_{\mathrm{R}} = \theta_{\mathrm{L}} = \theta_{\mathrm{R}} = 0. \end{aligned} \qquad (50)$$

In this example, the evolution of $E_n^{\mathbf{a}}$'s are very simple. In fact the evolution equations decouple and one has

$$\begin{aligned} \dot{E}_n^{\mathbf{a}} &= 2\mu E_n^{\mathbf{a}}, \quad 0 < n < L+1, \\ \dot{E}_{L+1}^{\mathbf{a}} &= 0, \\ E_0^{\mathbf{a}} &= 1. \end{aligned} \qquad (51)$$

The second example is less trivial. Let H satisfy (23), that is the system has left-right symmetry. Also let the covector $\mathbf{a}$ be

$$\mathbf{a} = (1, \xi, 0), \qquad (52)$$

and $\lambda = 0$. From the first (or third) equations of (18), one can find μ, ν, and θ in terms of the rates and the parameter ξ. The remaining equations are linear equations for rates. However, we have inequalities as well, namely the rates should be nonnegative. A specific class of the solutions can be obtained as

$$H = \begin{pmatrix} 0 & b & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & D_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & D_3 & 0 & 0 & r & \xi g & \xi(g+q) & \xi(g+q) \\ 0 & 0 & 0 & D_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g & f & 0 & D_6 & q & 0 & 0 \\ 0 & 0 & \xi g & c & 0 & \xi(g+q) & D_7 & r & \xi(g+q) \\ 0 & f & q & 0 & 0 & 0 & g & D_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 & u & D_9 \end{pmatrix},$$
$$(53)$$

where each diagonal element is minus the sum of the other elements in its column, and the following relations hold between the rates.

$$(1-\xi)b = \xi[c + f + (1-\xi)d],$$
$$(1-\xi)(g+q) = f + (1-\xi)d,$$
$$\xi u = (1-\xi)r. \qquad (54)$$

It is clear that for $0 < \xi < 1$, one can always find rates satisfying (54). With these rates, one has

$$\mu = -(1+\xi)(g+q),$$
$$\nu = g + q,$$
$$\theta = \xi(g+q). \qquad (55)$$

6 Concluding remarks

Among the aims of investigating reaction-diffusion systems is to find as many n-point functions (of number operators) as possible. There are systems for which one can find certain classes of these correlators.

The empty interval method was first introduced to investigate the probability of finding n neighboring empty sites, for systems consisting of one species, with nearest-neighbor interactions. One can generalize this method, in several aspects. One way is to consider systems with more-than-two-site interactions. Another way is to consider multi-species systems, and ask for the expectation of the product of certain linear combinations of the number operators. While this does not (generally) give the densities of each specific kind of particles, it does give a certain combination of these densities.

What was introduced here, was a set of constraints the reaction-diffusion systems should satisfy, in order that the system be solvable through the generalization of the empty interval method. For example, in a system consisting of particles of one kind (with nearest-neighbor interaction), there are 12 independent reaction-rates. If one demands that the system be left-right symmetric, the number of independent rates is reduced to 7. Among these,

there exists a 5-parameter family, the models of which are solvable through the generalized empty interval method. The classification of this family, and some examples, were discussed in Section 3. An interesting problem may be to classify the models satisfying the solvability conditions for multi-species systems.

7 Appendix

We want to solve the inequalities (28), for a given value of ξ. It is seen that changing the signs of ξ, ν, and θ simultaneously, while keeping the signs of λ, μ, and Λ fixed, does not change the inequalities. So it is sufficient to solve the inequalities for nonnegative ξ.

First consider the case $\mu = 0$. If $\mu = 0$, and $\xi = 1$, then one arrives at
i)

$$\xi = 1, \quad \mu = \theta = 0, \quad \nu \le 0, \quad 4\nu \le \lambda \le 2\nu. \qquad (56)$$

Another case is $\mu = 0$, $\xi \ne 1$. One can consider the subcases $\xi = 0$ and $\xi \ne 0$, and show that in both subcases, $\Lambda = \theta = \nu = 0$, which means the Hamiltonian is zero. So, if $\xi \ne 1$, for any nontrivial solution, $\mu \ne 0$.

If $\mu \ne 0$, then it should be negative. One can divide the Hamiltonian by $-\mu$ (which is like taking $\mu = -1$). The inequalities (18) are then rewritten as

$$(1-\xi)[-(1-\xi)\Lambda^- + 2(1+\tau)] \ge 0,$$
$$(1-\xi)[(1+\xi)\Lambda^- + 2\tau] \ge 0,$$
$$(1+\xi)[-(1+\xi)\Lambda^+ + 2(1-\tau)] \ge 0,$$
$$(1+\xi)[(1-\xi)\Lambda^- - 2\tau] \ge 0,$$
$$(1+\xi)^2\Lambda - 2\xi(1-\tau) \ge 0,$$
$$(1-\xi)^2\Lambda + 2\xi(1+\tau) \ge 0,$$
$$-(1-\xi^2)\Lambda + 2\xi\tau + 2 \ge 0, \qquad (57)$$

where τ and $\Lambda^\pm$ are defined through (30). Now, two general cases occur. Either $0 < \xi < 1$, or $1 < \xi$. First, take $0 < \xi < 1$. Then, the inequalities (57) become

$$2\tau \ge -2 + (1-\xi)\Lambda^- =: F_1,$$
$$2\tau \ge -(1+\xi)\Lambda^- =: F_2,$$
$$F_3 := 2 - (1+\xi)\Lambda^+ \ge 2\tau,$$
$$F_4 := (1-\xi)\Lambda^+ \ge 2\tau,$$
$$2\tau \ge 2 - \frac{(1+\xi)^2}{\xi}\Lambda =: F_5,$$
$$2\tau \ge -2 - \frac{(1-\xi)^2}{\xi}\Lambda =: F_6,$$
$$2\tau \ge -\frac{2}{\xi} + \frac{1-\xi^2}{\xi}\Lambda =: F_7. \qquad (58)$$

To have solution for τ, F_3 and F_4 must be greater than or equal to F_1, F_2, F_5, F_6, and F_7. So, we have ten inequalities for Λ^- and Λ^+. The first four, coming from the

$(F_3, \ F_4) \geq (F_1, \ F_2)$, are

$$
\begin{aligned}
4 - (1+\xi)\Lambda^+ - (1-\xi)\Lambda^- &\geq 0, \\
2 + (1-\xi)(\Lambda^+ - \Lambda^-) &\geq 0, \\
2 - (1+\xi)(\Lambda^+ - \Lambda^-) &\geq 0, \\
(1-\xi)\Lambda^+ + (1+\xi)\Lambda^- &\geq 0.
\end{aligned} \tag{59}
$$

The solution to these is the interior of a tetragon $ABCD$. The coordinates of its vertices are

$$
A\left(-\frac{1-\xi}{1+\xi}, 1\right), \quad B\left(1, \frac{3+\xi}{1+\xi}\right), \quad C\left(\frac{3-\xi}{1-\xi}, 1\right),
$$
$$
D\left(1, -\frac{1+\xi}{1-\xi}\right),
$$

where the first coordinate is Λ^-, and the second is Λ^+.

There remains six other inequalities for Λ^- and Λ^+, to be satisfied. As all of the inequalities are linear, it is sufficient to check the inequalities on the vertices of the above tetragon. In fact, we have to compare the values of F_3 and F_4, with those of F_5, F_6, and F_7, at the points A, B, C, and D. Doing so, it is seen that the problem is only with F_5 at the point D; all other inequalities are satisfied. At the segments DA and CD, $F_4 \leq F_3$. So we have to solve the inequality $F_4 \geq F_5$. The line $F_4 = F_5$ passes through A, and intersects the segment CD at the point E:

$$
E\left(\frac{1 + 6\xi - 3\xi^2}{(1-\xi)(1+3\xi)}, -\frac{1 + 3\xi^2}{(1-\xi)(1+3\xi)}\right).
$$

So,
ii)

$$
0 < \xi < 1, \quad \mu < 0 \ (\mu = -1) \quad (\Lambda^-, \Lambda^+) \text{ inside the}
$$
$$
\text{tetragon } ABCE, \quad \tau \text{ satisfies (58)}. \tag{60}
$$

For the case $1 < \xi$, one can still use (57). But as $(1-\xi)$ is negative, the first two inequalities in (58) are reversed. So we have

$$
(F_1, \ F_2, \ F_3, \ F_4) \geq 2\tau \geq (F_5, \ F_6, \ F_7). \tag{61}
$$

From these, one should have

$$
(F_1, \ F_2, \ F_3, \ F_4) \geq (F_5, \ F_6, \ F_7), \tag{62}
$$

which are twelve inequalities for $\Lambda^\pm$. From $F_1 \geq F_6$ and $F_3 \geq F_5$, one obtains

$$
\begin{aligned}
(\xi - 1)\Lambda^+ - (\xi + 1)\Lambda^- &\geq 0, \\
-(\xi - 1)\Lambda^+ + (\xi + 1)\Lambda^- &\geq 0,
\end{aligned} \tag{63}
$$

respectively. So, one has

$$
(\xi - 1)\Lambda^+ = (\xi + 1)\Lambda^- =: \chi. \tag{64}
$$

This also means that

$$
F_1 = F_3 = F_5 = F_6 = 2\tau. \tag{65}
$$

From $F_1 = F_3$, for example, χ is obtained:

$$
\chi = \frac{\xi^2 - 1}{\xi}. \tag{66}
$$

It is easily seen that this satisfies (61). So, one arrives at iii)

$$
1 < \xi, \quad \mu < 0 \ (\mu = -1) \quad \tau = -\frac{\xi^2 + 1}{2\xi}, \quad \Lambda^\pm = \frac{\xi \pm 1}{\xi}. \tag{67}
$$

It is seen that here there is no allowed region for the rates, but a single point (apart from scaling) for each value of ξ.

Finally, there remains two other special cases. First, the case $\xi = 0$. In this case, the first four inequalities in (57) are not changed, and hence the first four inequalities in (58). (One should of course put $\xi = 0$ in them.) The vertices of the tetragon $ABCD$ are now

$$
A_0(-1, 1), \quad B_0(1, 3), \quad C_0(3, 1), \quad D_0(1, -1).
$$

In the remaining three inequalities of (57), there is no τ, and one reads

$$
0 \leq \Lambda \leq 2. \tag{68}
$$

It is easy to see that these are satisfied inside the square $A_0 B_0 C_0 D_0$. This square is in fact the same tetragon $ABCE$ at the limit $\xi \to 0$. (Note that in this limit, the points D and E tend to each other.) So, iv)

$$
\xi = 0, \quad \mu \neq 0 \ (\mu = -1), \quad (\Lambda^-, \Lambda^+) \text{ is inside}
$$
$$
\text{the square } A_0 B_0 C_0 D_0. \ \tau \text{ satisfies (57)}. \tag{69}
$$

Finally, in the case $\xi = 1$, the first two inequalities in (57) become identities, and the sixth and seventh become identical to each other. So one has four independent inequalities:

$$
\begin{aligned}
-\Lambda^+ + 1 - \tau &\geq 0, \\
-\tau &\geq 0, \\
2\Lambda - (1 - \tau) &\geq 0, \\
1 + \tau &\geq 0.
\end{aligned} \tag{70}
$$

These give the allowed region for (Λ^-, Λ^+), as the region limited by the lines $A_1 B'$, the horizontal line passing through B', and the line passing through A_1 with the slope -1, containing the point $(1, 1)$, where

$$
A_1(0, 1), \quad B'(0, 2).
$$

Let us call this region $\mathbb{S}$. So,
v)

$$
\xi = 1, \quad \mu \neq 0 \ (\mu = -1), \quad (\Lambda^-, \Lambda^+) \text{ is inside } \mathbb{S}.
$$
$$
\tau \text{ satisfies (70)}. \tag{71}
$$

The cases i) to v) summarize the desired classification.

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